COMBINATORIAL FAMILIES ABOVE Θ

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1. Short Description of Research Agenda

This project will focus on studying the existence of independent families and their associates on cardinals κ in certain canonical inner models of the axiom of determinacy, establishing their existence or nonexistence. Studying in Vienna is important because it offers the ideal supervisors: Sandra Müller at TU Wien and Vera Fischer at the University of Vienna. Sandra Müller specializes in inner models of determinacy, while Vera Fischer is an expert in cardinal characteristics, from which independent families and associates arise. This project studies combinatorial families from cardinal characteristics in inner models of determinacy; Müller and Fischer are ideal and perfectly complemented advisors.

2. General Goals

Cantor spent much of his life trying to resolve the continuum hypothesis (CH), which is the statement that every set of real numbers is either countable or in bijection $\mathbb{R}$. Without resolution, Hilbert put it at the top of his list of ten problems presented at the Paris conference of the International Congress of Mathematicians in 1900. Then in 1939 Kurt Gödel proved that the continuum hypothesis cannot be refuted from the standard Zermelo–Frankel axioms of set theory (ZF) [Göd39]. Then in 1964 Paul Cohen proved that the ZF axioms cannot prove CH[Coh63].

Associated to the problem of the continuum hypotheses are cardinal characteristics, cardinal numbers which ZF proves are uncountable, but does not prove that they are the cardinality of $\mathbb{R}$. Cardinal characteristics often arise as the cardinality of a set of reals with some combinatorial properties, and these families are interesting in their own right. My research focuses on studying these sets with certain combinatorial properties, particularly their existence (or lack thereof) under the axiom of determinacy and its strengthenings.

3. Detailed Description of Research Problem

3.1. **Cardinal Characteristics.** While we cannot prove or refute CH from ZF, further analysis of the continuum problem leads to a study of certain kinds of sets of reals which *could* serve as witnesses to the failure of the continuum hypothesis, such things are called cardinal characteristics. While there is no formal definition of what a cardinal characteristic is, they typically take the form of "the minimal cardinality of a set of reals with some maximality property." The maximality property leads to our ability to prove that such characteristics are uncountable, but the fact that it is the minimal cardinality of such a set causes our inability to prove that cardinal characteristics have the same cardinality as $\mathbb{R}$. Here is a relatively short list of examples of cardinal characteristics:

(1) $\mathscr{A} \subseteq [\omega]^\omega$ is said to be almost–disjoint if for any $x, y \in \mathscr{A}$, $x \cap y$ is finite. $\mathscr{A}$ is maximally almost–disjoint (mad) if it is almost–disjoint and whenever $\mathscr{A}' \supseteq \mathscr{A}$ is almost–disjoint, then $\mathscr{A}' = \mathscr{A}$. Then the almost disjointness number, $\mathfrak{a}$, is the minimum cardinality of a mad family.

$$\mathfrak{a} = \min\{|\mathscr{A}| : \mathscr{A} \subseteq [\omega]^\omega \text{ is mad}\}$$

(2) $\mathscr{F} \subseteq [\omega]^\omega$ is said to be independent if whenever $X, Y \subseteq \mathscr{F}$ are finite and disjoint then $\bigcap X \setminus \bigcup Y$ is infinite. $\mathscr{F}$ is a maximally independent family (a mif) if it is independent and whenever $\mathscr{F}' \supseteq \mathscr{F}$ is independent then $\mathscr{F} = \mathscr{F}'$. Then the independence number is the minimal cardinality of a mif.

$$\mathfrak{i} = \min\{|\mathscr{F}| : \mathscr{F} \subseteq [\omega]^\omega \text{ is a mif}\}$$

(3) The covering number for Lebesgue measure, $\mathrm{cov}(\mathscr{L})$ is the minimal cardinality of a set $\mathscr{A}$ of measure zero sets whose union covers the real line, that is, such that $\bigcup_{X \in \mathscr{A}} X = \mathbb{R}$.

(4) The covering number for Baire category, $\mathrm{cov}(\mathscr{B})$ is the minimal cardinality of a set $\mathscr{A}$ of meagre sets whose union covers the real line, that is, such that $\bigcup_{X \in \mathscr{A}} X = \mathbb{R}$.

See [Bla09] for a larger catalog of examples. ZFC does not determine a linear order on the cardinal characteristics though in any model of ZFC, the relationship is linear. Classically, popular questions were about obtaining the consistency of certain linear relationships in the characteristics, e.g. $\mathsf{Con}(\mathsf{ZFC} + \mathfrak{b} < \mathfrak{a} < \mathfrak{s} < \mathfrak{i})$ remains open to the best of my knowledge [BF11].

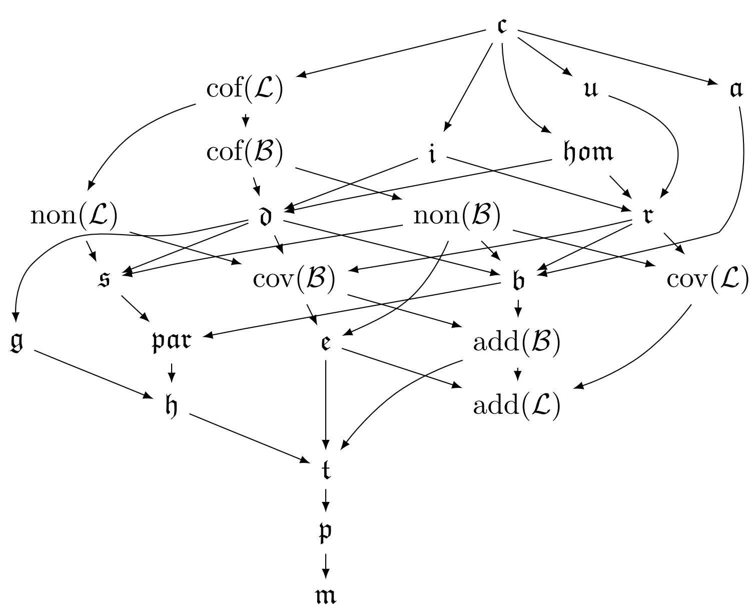

FIGURE 1. A Hasse diagram of many of the commonly studied characteristics, an arrow $x \to y$ is to mean $\mathsf{ZFC} \vdash y \leq x$

The use of the axiom of choice (AC) is implicit in the definition of all characteristics as writing "the minimum cardinality of a set with property P" assumes that the cardinal structure of the universe of sets is a well–ordering. However, for some characteristics the axiom of choice plays a second crucial role; for some of these characteristics the existence of a set with property P depends on choice. For example, without choice, there is a set of measure zero sets whose union is the entire real line, namely, $\mathscr{A} = \{\{x\} : x \in \mathbb{R}\}$, however, you cannot write down an example of a mif in ZF alone. Then a natural question to ask if the existence of a mif is consistent with $\mathsf{ZF} + \mathsf{AD}$ or $\mathsf{ZF} + \mathsf{AD}^+$, it turns out there are no mif's on ω if one assumes AD^+, this is recent unpublished work.

3.2. **Determinacy.** The most popular alternative to AC is the axiom of determinacy (AD) and its strengthening AD^+ which are conjectured to be equivalent. To each $A \subseteq \mathbb{R}$ we associate a two-player game G_A where players take turn playing integers so that at stage $i \in \omega$, player I plays n_{2i} and player II plays n_{2i+1}.

I	n_0		n_2		n_4			
								$\dots$
II		n_1		n_3		n_5		

Player I wins if the real $i \mapsto n_i$ belongs to A and otherwise II wins. A winning strategy for a player is a function $\sigma : \omega^{<\omega} \to \omega$, for player I it is such that if at all stages i, I plays $\sigma(\langle n_0, \dots, n_{2i-1}\rangle)$ then player I wins. Similarly for II. A game G_A is determined if one of the two players has a winning strategy. The axiom of determinacy is the statement that for all $A \subseteq \mathbb{R}$, G_A is determined. It is well known that AC and AD are incompatible, AD implies very strong regularity properties for sets of reals, consequences of AD include that all sets of reals: are Lebesgue measurable, have the property of Baire, have the perfect set property, are countable or non–well orderable. While AC is an axiom that affects the structure of all sets, AD has a limit. There is a cardinal Θ below which AD determines a lot of structure.

$$\Theta = \sup\{\xi \in \mathsf{On} : \exists f : \mathbb{R} \twoheadrightarrow \xi\}$$

This is due to the fact that while AC is a global statement, AD is a local statement about $\mathbb{R}$, so AD cannot determine global information. If one has a question about structure above Θ, one has to look at particular models of determinacy, e.g. $L(\mathbb{R})$, should it be a model of AD. See [Jec03] or [Kan03] for more detail.

3.3. **Almost disjoint and independent families under AD.** For any cardinal κ and an ideal $\mathscr{I}$ on κ we can define the notion of an $\mathscr{I}$–almost disjoint family and an $\mathscr{I}$–independent family. If $\mathscr{A}, \mathscr{F} \subseteq \mathscr{I}^+$ then we will say $\mathscr{A}$ is $\mathscr{I}$–almost disjoint if for any $x, y \in \mathscr{A}$, $x \cap y \in \mathscr{I}$ and $\mathscr{F}$ is $\mathscr{I}$–independent if for all finite disjoint $X, Y \subseteq \mathscr{F}$, $\bigcap X \setminus \bigcup Y \in \mathscr{I}^+$. Then we say $\mathscr{A}$ is $\mathscr{I}$–mad if it is $\mathscr{I}$–almost disjoint and if whenever $\mathscr{A}' \supseteq \mathscr{A}$ is $\mathscr{I}$–almost disjoint, then $\mathscr{A} = \mathscr{A}'$, similarly, $\mathscr{F}$

is an $\mathscr{I}$–mif if whenever $\mathscr{F}' \supseteq \mathscr{F}$ is $\mathscr{I}$–indepenent, then $\mathscr{F} = \mathscr{F}'$. Under AD^+, $\mathscr{I}$–mad families are well understood for κ with $\mathrm{cf}(\kappa) < \Theta$.

This direction of research is still very much in its infancy. In 2018 Norwood and Neeman showed that under AD^+, there are no mad or happy families on ω [NN18]. In 2019 Schrittesser and Törnquist sharpened this result for mad families weakening the hypothesis of AD^+ [ST19]. Then, Chan, Jackson, and Trang generalized this to all cardinals $\kappa < \Theta$ for various ideals and showed that in $L(\mathbb{R})$, for κ of cofinality at least Θ, that there are mad families on κ. More precisely, for the ideals

$$\mathscr{B}(\kappa) := \{X \subseteq \kappa : (\exists \xi < \kappa)(X \subseteq \xi)\} \qquad \text{and} \qquad [\kappa]^{<\kappa} := \{X \subseteq \kappa : |X| < \kappa\}$$

then the following theorem is due to Chan, Jackson, and Trang [CJT24].

Theorem 1 . Assume AD^+ and $\kappa \geq \omega_1$ is a cardinal then:

(a) » if $\mathrm{cf}(\kappa) = \omega$, then there are no $\mathscr{B}(\kappa)$–mad families on κ.
» if $\mathrm{cf}(\kappa) \in [\omega_1, \Theta)$, then there are no $\mathscr{B}(\kappa)$–mad families nor any $[\kappa]^{<\kappa}$–mad families.

(b) Assume $V = L(\mathbb{R})$ and $\mathrm{cf}(\kappa) \geq \Theta$, then there are $\mathscr{B}(\kappa)$–mad families and $[\kappa]^{<\kappa}$–mad families.

Under the supervision of Nam Trang I have been working toward similar results, though the story for independent families is more complicated. The combinatorics of independent families requires that one manages arbitrary boolean combinations, whereas the combinatorics of almost disjoint families only requires managing pairwise intersections. In unpublished work, we have the following:

Theorem 2 . Assume AD^+. Then:

» if κ is a cardinal of countable cofinality, then there are no $\mathscr{I}$–mifs for any $\mathscr{I}$ with $\mathscr{B}(\kappa) \subseteq \mathscr{I} \subseteq [\kappa]^{<\kappa}$.

» if κ is a cardinal with $\mathrm{cf}(\kappa) \in [\omega_1, \Theta)$, then there are no well–orderable $\mathscr{I}$–mifs on κ for either $\mathscr{I} = \mathscr{B}(\kappa)$ or $\mathscr{I} = [\kappa]^{<\kappa}$.

» if $\kappa < \Theta$ has uncountable cofinality and $\mathscr{F}$ is an $\mathscr{I}$–independent family on κ which is maximal, then there is an independent family $\mathscr{G}$ on ω with $|\mathscr{G}| = |\mathbb{R}|$ and an injection $\langle E_i : i \in \omega\rangle$ of ω into $\wp(\kappa) \setminus \mathscr{I}$ so that the map $\Phi : \mathscr{G} \to \mathscr{F}$ by

$$\Phi(x) = \bigcup_{i \in x} E_i$$

is well defined.

We have also studied other combinatorial structures coming from the theory of cardinal characteristics, though the most care was given to independent families due to their intricacies. Nothing is known about these kinds of families on cardinals κ with $\mathrm{cf}(\kappa) \geq \Theta$ in models with nontrivial structure above Θ, like the "Chang–type" models.

3.4. **Models of Determinacy.** For cardinals κ with $\mathrm{cf}(\kappa) \geq \Theta$, we cannot use AD^+ alone to analyze the structure of $\wp(\kappa)$, we need to look at particular models of determinacy. The consistency strength of $\mathsf{ZF} + \mathsf{AD}$ is strictly above ZFC, so one cannot prove in ZFC that the following models are (or even can be) models of AD, though assuming sufficiently strong large cardinal hypotheses they consistently can be models of determinacy and throughout we will assume that they are, see [ST24], [MS25], and [Woo21] to see the exact large cardinal hypotheses to achieve AD in the following. The most typical model of AD is Solovay's model, $L(\mathbb{R})$, the smallest model of ZF containing $\mathbb{R}$ and the ordinals, typical models of $\mathsf{AD}_{\mathbb{R}}$ (a strengthening of AD where players are allowed to play arbitrary real numbers, not integers alone) are of the form $L(\wp(\mathbb{R}))$. Though there are even further strengthenings of $\mathsf{AD}_{\mathbb{R}}$, namely AD + all sets of reals are universally Baire, which cannot have models of this typical type [Mü]. Examples of models with non-trivial structure above Θ are the Chang–type models, examples include $L(\mathsf{On}^\omega)$, $L(\mathsf{On}^\omega, \Gamma^\infty)$, and $L(\mathsf{On}^\omega)[\langle \mu_\alpha : \alpha \in \mathsf{On}\rangle]$ where μ_α is the club filter on $\wp_{\omega_1}(\alpha^\omega)$ and Γ^∞ is the collection of all $A \subseteq \mathbb{R}$ which are universally Baire. The goal for the project is to examine such models and establish the existence of maximally independent families and associated families on cardinals κ with cofinality above the Θ of the model.

4. Methodological Considerations

In the case of standard models of the from $L(\mathbb{R})$ the core set of tools used to analyze structure above Θ is how $L(\mathbb{R})$ relates to its inner models of choice. Namely Woodin's result that $L(\mathbb{R})$ is a symmetric collapse extension of $\mathsf{HOD}^{L(\mathbb{R})}$ [Cha19]. One can then use AC in $\mathsf{HOD}^{L(\mathbb{R})}$ to obtain structure and use Woodin's result to ensure the structure has certain properties in $L(\mathbb{R})$. A similar strategy with a similar set of tools can be used to analyze $L(\wp(\mathbb{R}))$ to achieve similar results. Each of the Chang–type models need to be handled separately, each having their own intricacies, though the HOD analyses of these models lead to tools of similar functionality. These inner models are closely related to hod mice which come from their HOD analyses, and this relationship will give us the tools we need to lift structure out of inner models of choice into the Chang–type models of interest [Mü] [GMS23].

5. Workflow

I expect to spend the first month working on studying the Chang model $L(\mathsf{On}^\omega)$ and establishing results about the existence or non-existence of mif's (and associates) on κ with $\mathrm{cf}(\kappa) \geq \Theta$. Then in months two and three move onto the more complicated model $L(\mathsf{On}^\omega, \Gamma^\infty)$ and if time permits $L(\mathsf{On}^\omega)[\langle \mu_\alpha : \alpha \in \mathsf{On}\rangle]$. Regularly meeting with Sandra Müller will be of crucial import to the project as she has expertise in hod mice and how these models are related to their hod mice. She is the perfect teacher to guide me through the development of the tools necessary to prove the conjectures of the next section. In the last few years, Vera Fischer has been studying similar generalizations of independent families and has had success using tools somewhat similar to Woodin's result in her work [FM22]. Meeting with Fischer to gain insight on how to more effectively use the tools developed with Müller will allow for the project to go further and possibly exceed the conjectures below.

6. Relevance and Expected Results

The expected outcome of the project is that the following conjectures will be shown to be true.

Conjecture 3 . Assume $V = L(\mathsf{On}^\omega) + \mathsf{AD}$ and κ is a cardinal with $\mathrm{cf}(\kappa) > \Theta$. Then there are $\mathscr{I}$–mad families and $\mathscr{I}$–mif's for both $\mathscr{I} = \mathscr{B}(\kappa)$ and $\mathscr{I} = [\kappa]^{<\kappa}$.

Conjecture 4 . Assume $V = L(\mathsf{On}^\omega, \Gamma^\infty) + \mathsf{AD}$ and κ is a cardinal with $\mathrm{cf}(\kappa) > \Theta$. Then there are $\mathscr{I}$–mad families and $\mathscr{I}$–mif's for both $\mathscr{I} = \mathscr{B}(\kappa)$ and $\mathscr{I} = [\kappa]^{<\kappa}$.

If time permits I plan to examine associates of the independent families which also come from the theory of cardinal characteristics in the same models. The general expectation is that for κ with $\mathrm{cf}(\kappa) > \Theta$ that there will be mif's, mad families, well–orderable reaping families etc. on κ. Having a better understanding of such families gives us a better understanding of the structure of $\wp(\kappa)$ in these models. With my background in tackling these kinds of questions in more standard models and below Θ, and with Sandra Müller's expertise in the structure of these non-trivial models I am confident that with her guidance we will resolve the existence of such families.

7. Retention Period

There is no retention period for this project.

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